

A Fluid Mechanics Primer

ARC Centre of Excellence for Climate Extremes

2020 Winter School

The 2020 CLEEx Winter School will focus on atmosphere and ocean dynamics, with lectures and labs covering topics of fundamental importance to fluid motions in both the atmosphere and ocean.

This primer is designed to ensure that participants have sufficient background knowledge of fluid mechanics. In general, lecturers will assume that students have familiarity with the equations and concepts described below.

There are three ways for you to assimilate this material:

1. Read the following pages. If you understand it all, then you will be okay.
2. If you are broadly familiar with the material in the primer but would like a refresher, sections 2-5 each have an accompanying video primer (provided by Prof Andy Hogg) which will expand upon and explain the steps to reach each equation and the importance of each equation.
3. Most of this material is also contained in Vallis' textbook:

Vallis, G. K. (2017). *Atmospheric and Oceanic Fluid Dynamics: Fundamentals and Large-Scale Circulation*. 2nd ed., **Cambridge Univ. Press**. 966pp.

Specific chapter numbers that you should be familiar with are noted below. This textbook is the approximate level which your lecturers will be aiming for.

Good luck and try to have fun!

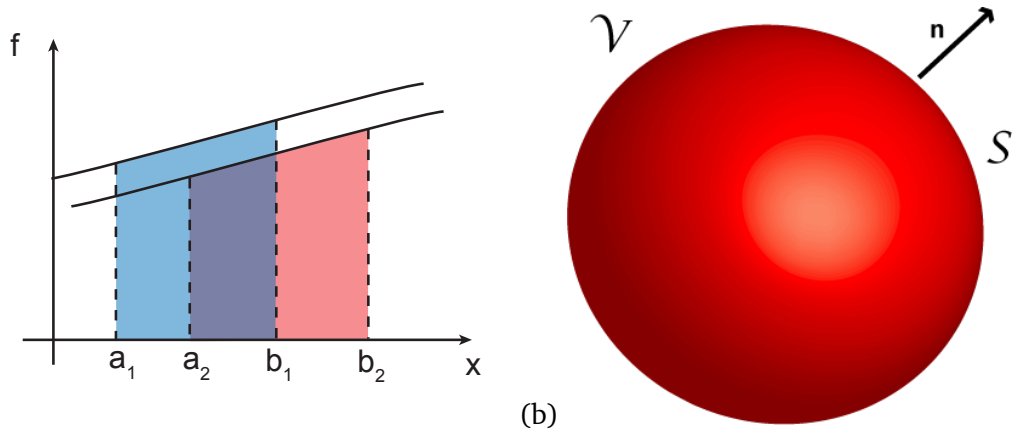


Figure 1: (a) A demonstration of Leibniz's rule in one dimension; (b) a schematic of our control volume.

1 Preliminaries

1.1 Mathematical Preliminaries

1.1.1 Leibniz's Theorem

Leibniz's Theorem states that:

$$\frac{d}{dt} \left(\int_{a(t)}^{b(t)} f(x, t) dx \right) = \int_{a(t)}^{b(t)} \frac{\partial f}{\partial t} dx + f(b, t) \frac{db}{dt} - f(a, t) \frac{da}{dt},$$

as shown schematically in figure 1. Leibniz's Theorem can be extended to 3 dimensions:

$$\frac{d}{dt} \left(\int_{\mathcal{V}(t)} f(\mathbf{x}, t) d\mathcal{V} \right) = \int_{\mathcal{V}(t)} \frac{\partial f}{\partial t} d\mathcal{V} + \int_{S(t)} f \mathbf{u} \cdot \hat{\mathbf{n}} dS,$$

where $\mathcal{V}(t)$ is a three-dimensional volume moving in time, and \mathbf{u} is the local velocity and $\hat{\mathbf{n}}$ is the local outward unit normal of the boundary of $\mathcal{V}(t)$.

1.1.2 Gauss' Theorem

Gauss' theorem, applied to an arbitrary vector field \mathbf{v} is

$$\int_{\mathcal{V}} \nabla \cdot \mathbf{v} d\mathcal{V} = \int_S \mathbf{v} \cdot \hat{\mathbf{n}} dS.$$

1.2 Lagrangian versus Eulerian Coordinate Systems (Vallis, 2017 §1.1)

Lagrangian coordinates describe a system where we define a fluid *particle* by its "initial" position $\mathbf{x}_0 = (x_0, y_0, z_0)$.

- Position:

$$\mathbf{x} = [x, y, z] \equiv \mathbf{x}(\mathbf{x}_0, t)$$

- Velocity:

$$\mathbf{u} = [u, v, w] \equiv \left. \frac{\partial \mathbf{x}}{\partial t} \right|_{\mathbf{x}_0}$$

- Acceleration:

$$\mathbf{a} \equiv \left. \frac{\partial^2 \mathbf{x}}{\partial t^2} \right|_{\mathbf{x}_0}$$

The Lagrangian Coordinate System is elegant, but conceptually difficult – therefore we tend to use Eulerian Coordinates, where:

- Position $\mathbf{x} = [x, y, z]$ is fixed in space.

- Velocity:

$$\mathbf{u} = [u, v, w] \equiv \mathbf{u}(\mathbf{x}, t)$$

- Define the **total time derivative** using Lagrangian notation:

$$\frac{D}{Dt} \equiv \left. \frac{\partial}{\partial t} \right|_{\mathbf{x}_0}$$

- e.g., scalar temperature, $T(\mathbf{x}, t)$. Apply the chain rule:

$$\left. \frac{\partial T}{\partial t} \right|_{\mathbf{x}_0} = \frac{\partial T}{\partial x} \underbrace{\left. \frac{\partial x}{\partial t} \right|_{\mathbf{x}_0}}_{=u} + \frac{\partial T}{\partial y} \underbrace{\left. \frac{\partial y}{\partial t} \right|_{\mathbf{x}_0}}_{=v} + \frac{\partial T}{\partial z} \underbrace{\left. \frac{\partial z}{\partial t} \right|_{\mathbf{x}_0}}_{=w} + \frac{\partial T}{\partial t} \underbrace{\left. \frac{\partial t}{\partial t} \right|_{\mathbf{x}_0}}_{=1}$$

... giving ...

$$\frac{DT}{Dt} \equiv \frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T.$$

1.2.1 The Control Volume

The equations of motion are derived by considering a **control volume** either fixed in space (Eulerian) or made up of a fixed set of fluid particles (Lagrangian). The control volume $\mathcal{V}(\mathbf{x}_0, t)$ is encapsulated by surface $\mathcal{S}(\mathbf{x}_0, t)$, with unit normal $\hat{\mathbf{n}}$.

1.3 Continuum Hypothesis

The *Continuum Hypothesis* states that we can associate with any volume of fluid, no matter how small, the macroscopic properties that we associate with the fluid in bulk. This ensures that properties of the fluid at a given location are well-defined. For example, for the density ρ we have,

$$\rho \equiv \lim_{\Delta \rightarrow 0} \left(\frac{M}{V} \right),$$

where M and V are the mass and volume, respectively, of a control volume of scale Δ . However, real fluids are made of molecules, and at some scale this hypothesis must therefore break down. Nevertheless, if there exists a scale L_2 such that $L_1 \ll L_2 \ll L_3$, where L_1 is the scale at which molecular effects become important and L_3 is the scale at which fluid properties vary with the flow, we may define the density at a point to be,

$$\rho \equiv \lim_{\Delta \rightarrow L_2} \left(\frac{M}{V} \right).$$

With this definition, the fluid properties may be treated as continuous, and variations on scales much smaller than L_2 due to molecular effects may be ignored. Note that

- The continuum hypothesis breaks down at very small scales;
- It is generally justified by experiments;
- Fluid mechanics relies upon this hypothesis.

1.4 Dry thermodynamics of a perfect gas (Vallis, 2017 §1.4-§1.5)

A perfect (or ideal) gas is one in which the individual molecules are sufficiently far apart that the short range force that acts between them can be ignored, and consequently collisions between molecules are perfectly elastic. Laboratory experiments have shown that for a perfect gas, the temperature T , pressure p and density ρ are related by

$$p = \rho RT,$$

where $R = c_p - c_v = 287 \text{ J kg}^{-1} \text{ K}^{-1}$ is the gas constant for dry air, $c_p = 1004 \text{ J kg}^{-1} \text{ K}^{-1}$ is the specific heat at constant pressure, and $c_v = 717 \text{ J kg}^{-1} \text{ K}^{-1}$ is the specific heat at constant volume. The first law of thermodynamics expresses the principle of energy conservation and is commonly written as

$$c_v \frac{DT}{Dt} + p \frac{D\alpha}{Dt} = \dot{Q}.$$

The first term in this equation represents the rate of change of internal energy per unit mass $c_v T$. The next term is the rate of working by the fluid per unit mass where $\alpha \equiv \rho^{-1}$ is the specific volume. This term describes the conversion between thermal and mechanical energy. The final term represents the rate of diabatic heating per unit mass due to radiation, conduction and latent heat release.

Using the two above equations, it can be shown that

$$\frac{Ds}{Dt} = \frac{\dot{Q}}{T},$$

where $s = c_p \ln T - R \ln p + \text{constant}$, defines the specific entropy. Two useful alternative expressions for the specific entropy are

$$s = c_v \ln p - c_p \ln \alpha + \text{constant},$$

and

$$s = c_p \ln \theta,$$

where $\theta = T(p_r/p)^\kappa$ is the potential temperature, p_r is a reference pressure usually taken to be 1000 mb, and $\kappa = R/c_p$. An adiabatic process is one for which $\dot{Q} \equiv 0$. In this case we get $Ds/Dt = 0$, indicating that the specific entropy (or equivalently potential temperature) is conserved following fluid parcels.

With the above preliminaries in hand we outline the derivation of equations of motion for a fluid in the following sections. These equations can be derived from integral conservation laws to give partial differential equations for fluid motion (e.g., the Navier-Stokes equations).

2 Conservation of Mass (Vallis, 2017 §1.2)

The relationship between mass M and density ρ within our control volume is:

$$M = \int_{\mathcal{V}(t)} \rho \, d\mathcal{V}.$$

Assume that mass M is constant, i.e.,

$$\frac{d}{dt} \int_{\mathcal{V}(t)} \rho \, d\mathcal{V} = 0.$$

Apply Leibniz rule in 3D over the control volume:

$$\int_{\mathcal{V}(t)} \frac{\partial \rho}{\partial t} \, d\mathcal{V} + \int_{S(t)} \rho \mathbf{u} \cdot \hat{\mathbf{n}} \, dS = 0,$$

and then use Gauss' Theorem to get

$$\int_{\mathcal{V}(t)} \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) \right] \, d\mathcal{V} = 0.$$

This equation applies even in the limit $\mathcal{V} \rightarrow 0$ and, therefore, it should hold *at every point*, i.e.,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0.$$

We now break apart the second term to write an equation in terms of the local rate of change, the advective and the divergent contribution:

$$\begin{aligned} \frac{\partial \rho}{\partial t} &+ \mathbf{u} \cdot \nabla \rho &+ \rho \nabla \cdot \mathbf{u} &= 0, \\ \text{Rate of change} &+ \text{Advection} &+ \text{Divergence} &= 0. \end{aligned}$$

Using the definition of the the total time derivative, we can rewrite the above as

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{u} = 0.$$

This is the conservation of mass equation. Note that if fluid is incompressible (like the ocean is, almost) then

$$\frac{D\rho}{Dt} = 0, \quad \text{and hence} \quad \nabla \cdot \mathbf{u} = 0.$$

3 Advection–Diffusion equation

Start with Fick’s first law to represent diffusive processes:

- Flux of a quantity with concentration $C(x, t)$ is proportional to its gradient;

$$\text{Flux} \propto \kappa_c \frac{\partial C}{\partial x}.$$

- At a molecular level, Fick’s Law is a consequence of Brownian motion;
- Equivalent to Fourier’s law for heat conduction;
- More generally, if $C = C(\mathbf{x}, t)$, then

$$\text{Flux} \propto \kappa_c \nabla C.$$

Assume we have a concentration C of a substance which diffuses across our control volume boundary by Fick’s law:

$$\frac{d}{dt} \int_{\mathcal{V}(t)} C \, d\mathcal{V} = \int_{\mathcal{S}(t)} \kappa_c \nabla C \cdot \hat{\mathbf{n}} \, d\mathcal{S}.$$

Apply Leibniz rule in 3D over the control volume:

$$\int_{\mathcal{V}(t)} \frac{\partial C}{\partial t} \, d\mathcal{V} + \int_{\mathcal{S}(t)} C \mathbf{u} \cdot \hat{\mathbf{n}} \, d\mathcal{S} = \int_{\mathcal{S}(t)} \kappa_c \nabla C \cdot \hat{\mathbf{n}} \, d\mathcal{S},$$

and apply Gauss’ Theorem to give

$$\int_{\mathcal{V}(t)} \left[\frac{\partial C}{\partial t} + \nabla \cdot (C \mathbf{u}) \right] \, d\mathcal{V} = \int_{\mathcal{V}(t)} \nabla \cdot (\kappa_c \nabla C) \, d\mathcal{V}.$$

This equation applies even in the limit $\mathcal{V} \rightarrow 0$. Therefore it applies at every point:

$$\frac{\partial C}{\partial t} + \nabla \cdot (C \mathbf{u}) = \kappa_c \nabla^2 C,$$

where we have assumed that κ_c is constant. Applying the product rule to the 2nd term on the LHS,

$$\frac{\partial C}{\partial t} + \mathbf{u} \cdot \nabla C + C \nabla \cdot \mathbf{u} = \kappa_c \nabla^2 C,$$

which, for an incompressible fluid ($\nabla \cdot \mathbf{u} = 0$) can be reduced to

$$\frac{\partial C}{\partial t} + \mathbf{u} \cdot \nabla C = \kappa_c \nabla^2 C,$$

$$\frac{DC}{Dt} = \kappa_c \nabla^2 C.$$

This is the advection–diffusion equation.

4 Stress and the momentum equation (mostly in Vallis, 2017 §1.3)

4.1 Newton’s Second law

$$\mathbf{F} = m\mathbf{a} = \frac{d}{dt}[\text{Momentum}]$$

Forces include:

- Stress

$$\boldsymbol{\tau} = \underline{\boldsymbol{\sigma}} \cdot \hat{\mathbf{n}}$$

where $\underline{\boldsymbol{\sigma}}$ is called the *stress tensor* (discussed later).

- Body force (i.e., gravity, \mathbf{g}).
- Both of these forces act to change momentum, $\rho\mathbf{u}$.

4.2 Conservation of Momentum

Assume that momentum in the control volume is conserved, i.e.,

$$\frac{d}{dt} \int_{\mathcal{V}(t)} \rho\mathbf{u} \, d\mathcal{V} = \int_{\mathcal{V}(t)} \rho\mathbf{g} \, d\mathcal{V} + \int_{\mathcal{S}(t)} \underline{\boldsymbol{\sigma}} \cdot \hat{\mathbf{n}} \, d\mathcal{S}$$

We’ll play the same game once more: Let’s apply the Leibniz rule in 3D over the control volume:

$$\int_{\mathcal{V}(t)} \frac{\partial(\rho\mathbf{u})}{\partial t} \, d\mathcal{V} + \int_{\mathcal{S}(t)} (\rho\mathbf{u})\mathbf{u} \cdot \hat{\mathbf{n}} \, d\mathcal{S} = \int_{\mathcal{V}(t)} \rho\mathbf{g} \, d\mathcal{V} + \int_{\mathcal{S}(t)} \underline{\boldsymbol{\sigma}} \cdot \hat{\mathbf{n}} \, d\mathcal{S},$$

and also use Gauss’ Theorem on the RHS and expand the second term on the LHS using the product rule. Then we get:

$$\int_{\mathcal{V}(t)} \left[\frac{\partial(\rho\mathbf{u})}{\partial t} + \mathbf{u} \cdot \nabla(\rho\mathbf{u}) + (\rho\mathbf{u})\nabla \cdot \mathbf{u} \right] d\mathcal{V} = \int_{\mathcal{V}(t)} \left(\rho\mathbf{g} + \nabla \cdot \underline{\boldsymbol{\sigma}} \right) d\mathcal{V}.$$

You know the drill... This equation applies even in the limit $\mathcal{V} \rightarrow 0$. Therefore it should hold *at every point*:

$$\frac{D(\rho\mathbf{u})}{Dt} + \rho\mathbf{u}\nabla \cdot \mathbf{u} = \rho\mathbf{g} + \nabla \cdot \underline{\sigma}$$

. Apply product rule to the first term gives

$$\rho \frac{D\mathbf{u}}{Dt} + \mathbf{u} \left(\underbrace{\frac{D\rho}{Dt} + \rho\nabla \cdot \mathbf{u}}_{=0} \right) = \rho\mathbf{g} + \nabla \cdot \underline{\sigma}$$

. The bracketed term now cancels (conservation of mass!) to give

$$\rho \frac{D\mathbf{u}}{Dt} = \rho\mathbf{g} + \nabla \cdot \underline{\sigma}. \quad (1)$$

4.3 Stress

- Stress has units N/m²;
- Think of it as force per unit area – like pressure;
- Stress in a fluid is *usually* closely linked with the velocity gradient, i.e.,

$$\tau = \mu \frac{\partial u}{\partial y}$$

where μ is the coefficient of viscosity.

- $\nu \equiv \mu/\rho$ is defined as the kinematic viscosity – we'll use that later.
- Stress is the force applied across an area, or surface.
- This surface also has a *direction* (normal vector $\hat{\mathbf{n}}$).
- Define the *stress tensor*, $\underline{\sigma} = \sigma_{ij}$, to be acting in direction x_j on a surface with normal vector n_i .
- Write stress components as:

$$\sigma_{ij} = -p\delta_{ij} + \tau_{ij}$$

where p is pressure, τ_{ij} is the viscous stress tensor, and δ_{ij} is the Kronecker delta.

4.4 Euler's Equations

If $\tau_{ij} = 0$ (i.e., there is no viscous stress) then we obtain an inviscid momentum equation from Eq. (1), i.e.,

$$\frac{D\mathbf{u}}{Dt} = \mathbf{g} - \frac{\nabla p}{\rho}. \quad (2)$$

4.5 Navier–Stokes Equations

We assume that:

- The stress tensor is symmetric (this can be proven!)
- Viscous stress depends on velocity gradients
- Viscosity is constant
- Fluid is incompressible

Then

$$\tau_{ij} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

$$\underline{\tau} = \mu \begin{pmatrix} 2\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} & \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \\ \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} & 2\frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \\ \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} & \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} & 2\frac{\partial w}{\partial z} \end{pmatrix}$$

or

$$\underline{\sigma} = \begin{pmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{pmatrix} + \mu \begin{pmatrix} 2\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} & \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \\ \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} & 2\frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \\ \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} & \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} & 2\frac{\partial w}{\partial z} \end{pmatrix}$$

$$\nabla \cdot \underline{\sigma} = -\nabla p + \mu \begin{pmatrix} 2\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 w}{\partial x \partial z} \\ \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial y \partial x} + 2\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} + \frac{\partial^2 w}{\partial y \partial z} \\ \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 u}{\partial z \partial x} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 v}{\partial z \partial y} + 2\frac{\partial^2 w}{\partial z^2} \end{pmatrix}$$

Recall that we assumed incompressibility, so that continuity ($\nabla \cdot \mathbf{u} = 0$) gives

$$\nabla \cdot \underline{\sigma} = -\nabla p + \mu \nabla^2 \mathbf{u},$$

leading to the Navier Stokes equation:

$$\frac{D\mathbf{u}}{Dt} = \mathbf{g} - \frac{\nabla p}{\rho_0} + \nu \nabla^2 \mathbf{u}, \quad (3)$$

where ρ_0 is a constant reference density and we have used the Boussinesq approximation (see Vallis, 2006, §2.4). This is the fundamental equation used to solve many fluid mechanics problems.

5 Rotating reference frame (Vallis, 2017 §2.1)

5.1 Coriolis force

Consider a reference frame that is rotating at a rate Ω as shown in figure 2(a). A point

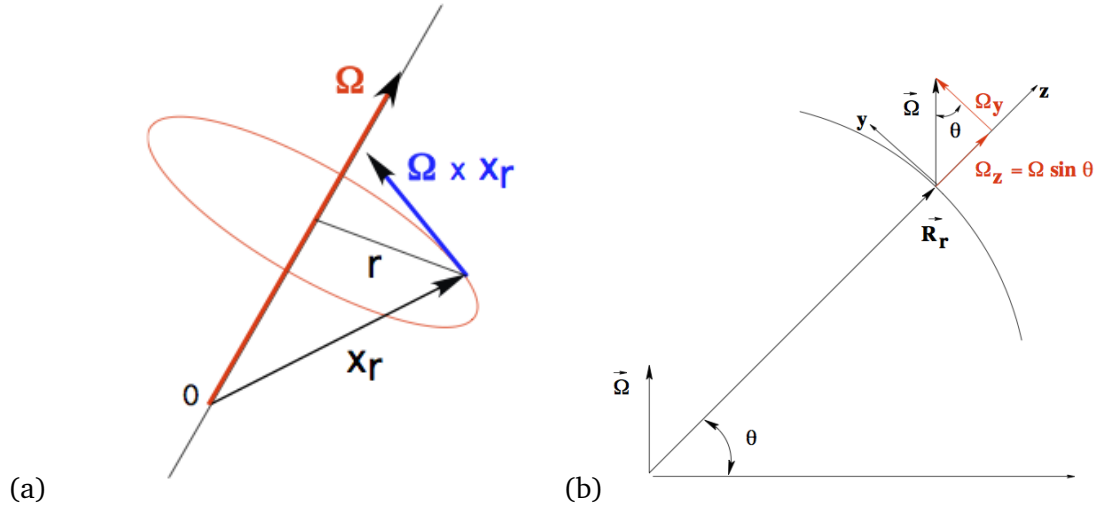


Figure 2: (a) Motion in a rotating reference frame; (b) components of $\boldsymbol{\Omega}$ on the surface of a rotating sphere.

P fixed at position \mathbf{x}_r in the rotating frame moves in a circle relative to a fixed frame (subscript f):

$$\frac{d_r \mathbf{x}_r}{dt} = 0; \quad \frac{d_f \mathbf{x}_r}{dt} = \boldsymbol{\Omega} \times \mathbf{x}_r$$

If P also moves relative to rotating frame, its velocity in fixed frame is

$$\mathbf{u}_f = \frac{d_f \mathbf{x}_r}{dt} = \frac{d_r \mathbf{x}_r}{dt} + \boldsymbol{\Omega} \times \mathbf{x}_r$$

This relation holds for any vector quantity. Hence velocity in the fixed frame satisfies

$$\begin{aligned} \frac{d_f \mathbf{u}_f}{dt} &= \frac{d_r \mathbf{u}_f}{dt} + \boldsymbol{\Omega} \times \mathbf{u}_f \\ &= \frac{d_r}{dt} \left[\frac{d_r \mathbf{x}_r}{dt} + \boldsymbol{\Omega} \times \mathbf{x}_r \right] + \boldsymbol{\Omega} \times \left[\frac{d_r \mathbf{x}_r}{dt} + \boldsymbol{\Omega} \times \mathbf{x}_r \right] \\ &= \frac{d_r^2 \mathbf{x}_r}{dt^2} + 2\boldsymbol{\Omega} \times \frac{d_r \mathbf{x}_r}{dt} + \boldsymbol{\Omega} \times \boldsymbol{\Omega} \times \mathbf{x}_r \\ &= \frac{d_r \mathbf{u}_r}{dt} + 2\boldsymbol{\Omega} \times \mathbf{u}_r + \boldsymbol{\Omega} \times \boldsymbol{\Omega} \times \mathbf{x}_r \end{aligned}$$

In other words, the acceleration in the fixed frame is given by the acceleration in the rotating frame plus two correction terms. These terms are the Coriolis acceleration and the centrifugal acceleration, respectively.

To apply this result to the Navier Stokes equation, we start with the equation in the fixed frame,

$$\frac{D_f \mathbf{u}_f}{Dt} = \mathbf{g} - \frac{\nabla p}{\rho_0} + \nu \nabla^2 \mathbf{u}_f,$$

and transform onto the rotating frame,

$$\frac{D_r \mathbf{u}_r}{Dt} + 2\boldsymbol{\Omega} \times \mathbf{u}_r = \mathbf{g} - \boldsymbol{\Omega} \times \boldsymbol{\Omega} \times \mathbf{x}_r - \frac{\nabla p}{\rho_0} + \nu \nabla^2 \mathbf{u}_f.$$

The centrifugal acceleration can be absorbed into the gravitational potential, so we can write (after dropping r subscripts):

$$\frac{D\mathbf{u}}{Dt} + 2\boldsymbol{\Omega} \times \mathbf{u} = -\nabla\Phi - \frac{\nabla p}{\rho_0} + \nu \nabla^2 \mathbf{u}, \quad (4)$$

where Φ is called the geopotential, which combines the gravitational potential, and a potential whose gradient is equal to the centrifugal acceleration. For simplicity, we will write $-\nabla\Phi = \mathbf{g}$.

5.2 Components of the rotating momentum equation

The ocean and atmosphere are shallow fluid layers, so that it is the local horizontal components of the Coriolis force which is important. The vertical component of $\boldsymbol{\Omega}$ at any location on the surface of a rotating sphere is $2|\boldsymbol{\Omega}| \sin \theta$, where θ is latitude (as per figure 2b). Therefore, we define

$$f = 2|\boldsymbol{\Omega}| \sin \theta$$

and rewrite the momentum equation as

$$\frac{D\mathbf{u}}{Dt} + f\hat{\mathbf{k}} \times \mathbf{u} = \mathbf{g} - \frac{\nabla p}{\rho_0} + \nu \nabla^2 \mathbf{u}.$$

Remember that the momentum equation has 3 components. We can unpack each component and the total time derivative by writing

$$\begin{aligned} \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} - fv &= -\frac{1}{\rho_0} \frac{\partial p}{\partial x} + \nu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right), \\ \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} + fu &= -\frac{1}{\rho_0} \frac{\partial p}{\partial y} + \nu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right), \\ \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} &= -g - \frac{1}{\rho_0} \frac{\partial p}{\partial z} + \nu \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right). \end{aligned}$$

Appendix A List of standard symbols

- ρ – Density
- T – Temperature
- $\mathbf{x} = [x, y, z]$ – Position
- $\mathbf{u} = [u, v, w]$ – Velocity

- t – Time
- $\hat{\mathbf{n}}$ – Unit normal
- C – Concentration
- κ_c – Molecular diffusion coefficient
- κ_T – Turbulent diffusion coefficient
- $\boldsymbol{\tau}$ – Stress (vector)
- $\underline{\boldsymbol{\sigma}}$ – Stress tensor
- μ – Coefficient of viscosity
- ν – Kinematic viscosity
- p – Pressure
- $\mathbf{g} = [0, 0, -g]$ – Acceleration due to gravity